Sequence :→ A sequence is a function whose domain is the set of all natural numbers ($\mathbb{N}$) whereas the range may be any set S.

Real Sequence :→ A real sequence is a function whose domain is the set of all natural number ($\mathbb{N}$) and range a subset of the set of real numbers ($\mathbb{R}$).

Range of a Sequence :→ The set of all distinct terms of a sequence is called its range.

"Sequence is denoted by $\{x_n\}$, where $n \in \mathbb{N}$."

Constant Sequence :→ A sequence $\{x_n\}$ defined by $x_n = c \in \mathbb{R}$. $\forall\, n \in \mathbb{N}$. is called a constant sequence.

$\{x_n\} = \{c, c, c, \ldots\}$ is a constant sequence with range $= \{c\}$.

Convergent Sequence :→ A sequence $\{a_n\}$ is said to be convergent if $\lim\limits_{n\to\infty} a_n$ is finite.

for Ex :→ $a_n = \dfrac{1}{2^n}$, Consider the sequence

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \frac{1}{2^n} \ldots$$

Here, n^{th} term of the sequence is $a_n = \dfrac{1}{2^n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0, \text{ which is finite.}$$

$\Rightarrow$ the sequence is Convergent.

Divergent Sequence :→ A sequence $\{a_n\}$ is said to be ~~convergent~~ divergent if $\lim_{n\to\infty} a_n$ is not finite.

i.e if $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$.

For Ex : Consider the sequence $\{n^2\}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = +\infty \Rightarrow$ the sequence is divergent.

(ii) Consider the sequence. $\{-2^n\}$.

$a_n = -2^n$, $\lim_{n\to\infty} -2^n = -\infty \Rightarrow$ Divergent.

Oscillatory Sequence :→ If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence.

Oscillatory sequence are of two types →

(i) A bounded sequence which does not converge is said to oscillate finitely.

For Ex :→ Consider the sequence $\{(-1)^n\}$.

$a_n = \{(-1)^n\}$.

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} = -1$$

$\lim_{n\to\infty} a_n$ does not exist. ⇒ the sequence does not converge.

Hence the sequence oscillate finitely.

(ii) An unbounded sequence which does not diverge is said to oscillate infinitely.

Monotonic Sequences :→ A sequence $\{a_n\}$ is said to be monotonically increasing if

$a_{n+1} \geq a_n \quad \forall \ n \in \mathbb{N}.$

i.e if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$

(ii) A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}.$

i.e $a_1 \geq a_2 \geq a_3 \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(iii) A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

(iv) A sequence $\{a_n\}$ is said to be strictly monotonically increasing if $a_{n+1} > a_n \quad \forall n \in \mathbb{N}.$

(v) A sequence $\{a_n\}$ is said to be strictly monotonically decreasing if $a_{n+1} < a_n \quad \forall n \in \mathbb{N}.$

Note :→ Every convergent sequence is bounded.

Q. Give an example of a monotonic increasing sequence which is convergent & divergent.

Soln. (i) Consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots \frac{n}{n+1}, \cdots$

Since $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \cdots$, the sequence is monotonic increasing.

Here, n^{th} term of the sequence is

$$a_n = \frac{n}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n\left[1+\frac{1}{n}\right]} = 1$$

which is finite $\Rightarrow$ sequence is convergent.

(ii) Consider the sequence $1, 2, 3, \ldots, n, \ldots$

Since, $1 < 2 < 3 < \ldots$, the sequence is monotonic increasing.

Here, $a_n = n$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = +\infty$

$\therefore$ the sequence is diverge to $+\infty$.

Q. Give an example of a monotonic decreasing sequence which is → (i) Convergent (ii) Divergent.

Sol. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

Since, $1 > \frac{1}{2} > \frac{1}{3} > \ldots$ the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

the sequence converges to zero.

(ii) Consider the sequence $-1, -2, -3, \ldots, -n, \ldots$

Since, $-1 > -2 > -3 > \ldots$, the sequence is monotonic decreasing

$$a_n = -n, \quad \lim_{n\to\infty} -n = -\infty \Rightarrow \text{diverge.}$$

(n^{th} term of sequence which is denoted by a_n)

Q. Discuss the convergence of the sequence $\{a_n\}$ where

(i) $a_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} \frac{\cancel{n}\left[1+\frac{1}{n}\right]}{\cancel{n}} = \lim_{n\to\infty} 1+\frac{1}{n} = 1, \text{ which is finite}$$

$\Rightarrow$ It is convergent.

Solved it.

(ii) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$.

Series :→ An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ in which each term is obtained from its preceding term by some definite law is called a series. This definite law is known as law of formation of the series $u_1, u_2, u_3 \cdots u_n, \cdots$ are respectively known as Ist, IInd, third, n^{th}, -- term of the series.

Finite Series :→ A series which contains a finite number of terms is called a finite series.

Thus $\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_r + \cdots + u_n$

is the finite series.

Infinite Series :→ A series which contains an infinite number of terms is called an infinite series

i.e $\sum_{r=1}^{\infty} u_r = u_1 + u_2 + \cdots + u_n + \cdots$

* [the sum of Ist n-term of the series is denoted by "S_n"]

Thus, $S_n = u_1 + u_2 + u_3 + \cdots + u_n$.

S_n is also called the n^{th} partial sum of $\sum_{n=1}^{\infty} u_n$

Positive Term Series :→ The series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is said to be a positive term series if every term of the series is positive.

Similarly, we can define a negative term series.

Alternating Series :→ If the term of a series are alternately positive and negative beginning with the first term then, the series is said to be an alternating series.

Thus, $u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1} u_n + \cdots$

is an alternating series.

Imp.

Convergence & Divergence of a Series :→

Let $\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ be an infinite series. Then

$$S_n = u_1 + u_2 + \cdots + u_n.$$

then (i) If $\lim_{n \to \infty} S_n =$ finite and unique quantity

then $\sum u_n$ is convergent.

(ii) If $\lim_{n \to \infty} S_n = +\infty$ or $-\infty$

then $\sum u_n$ is divergent.

(iii) $\lim_{n\to\infty} S_n$ is neither a finite unique quantity nor $+\infty$ or $-\infty$ then the series Σu_n is said to be oscillatory.

A series can oscillate finitely or infinitely.

i.e the series Σu_n is said to oscillate finitely if $\lim_{n\to\infty} S_n$ is a finite but not a unique quantity.

& if $\lim_{n\to\infty} S_n$ fluctuate b/we finite limits.

(ii) If $\lim_{n\to\infty} S_n$ fluctuate b/we $-\infty$ and $+\infty$ then Σu_n is oscillate infinitely.

Que Consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+$ ——

Sol^n Here, S_n (Sum of n^{th} term) $= \dfrac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$

[It is G.P Series; so Sum of n^{th} term is $S_n = \dfrac{a(1-r^n)}{1-r}$ Here, $a \to 1^{st}$ term i.e $a=1$]

$$S_n = 2\left(1-\frac{1}{2^n}\right)$$

[Common ratio $(r) = \frac{1}{2}$]

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right)$$

$= 2$ which is finite & unique

$\Rightarrow$ It is convergence.

Q. Consider the series, $1+2+3+4+5+ ——$

Here, $S_n = \frac{n(n+1)}{2}$ [It is A.P series. & Sum of n^{th} term is. $S_n = \frac{n}{2} + [2a + (n-1).d]$]

$\lim_{n \to \infty} S_n = +\infty \Rightarrow$ It is divergent. Here, $a=1$, $d=1$

(d = Common difference
a = 1^{st} term.)